Qu1

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(a) In $z = \tanh w = \dfrac{e^w - e^{-w}}{e^w + e^{-w}}$, try to write w in terms of z.

(b) Log is defined on $\mathbb{C} \setminus (-\infty, 0]$

$$\frac{1+z}{1-z} \neq -t \quad \text{where} \quad t \in [0, \infty)$$

Thus $z \notin (-1, 1)$

$\mathrm{Log}\,\frac{1+z}{1-z}$ is analytic on $\mathbb{C} \setminus (-1, 1)$

Qu2

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(a) The set is $2^i = e^{i\log 2} = e^{i[\ln 2 + i \arg 2]}$

$$= e^{-\arg 2}\cdot(\cos\ln 2 + i\sin\ln 2)$$

$$= \{e^{-2k\pi}(\cos\ln 2 + i\sin\ln 2) : k \in \mathbb{Z}\}$$

(b) The principal branch of $(z^2+1)^i$ is defined on $\Omega_0 \subset \mathbb{C}\setminus\{\pm i\}$ where $\mathrm{Log}(z^2+1)$ is defined, i.e., $z^2+1 \notin (-\infty, 0]$ or $z^2 \notin (-\infty, -1] \iff$

$z \notin \{iy : y \in [1,\infty) \text{ or } y \in (-\infty,-1]\}$

(c) Note that Ω_0 above does not have $2i \in \Omega_0$

Try another branch of log, e.g., Log_α

Choose a suitable α s.t. $2i \in \Omega_1$

Qu3

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(a) Do it by parametrizing L_1 and L_2

Then direct calculation is okay.

(b) $$\left| \int_C \frac{dz}{z+1} \right| \le \int_C \frac{|dz|}{|z+1|}$$

Note that $|z+1|$ is the distance between z and -1, while $z \in C$

Moreover, $\int_C |dz|$ is the arc length.

Qu4

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(a) By parametrization and direct calculation.

Note. Some people thought too simply and assumed $\frac{1}{z} \neq \bar{z}$ where z is not on the unique circle.

(b) You may fix the parametrization

Then use different branches of $z^{1/3}$.

There should be many branches, carefulness is needed to give 3 different answers.

Qu5

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(a) Choose an antiderivatives and apply independant of choices, and so use the initial and final points.

(b) Simply apply Cauchy-Goursat Theorem or Cauchy Theorem suitably.

(c) Verify that

$$\frac{P'(z)}{P(z)} = \frac{1}{z-z_1} + \cdots + \frac{1}{z-z_p}$$

Thus it can be easy to find the integral